Available online at www.sciencedirect.com
ScienceDirect
European Journal of Combinatorics 29 (2008) 746–759
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Completely compressible Bruhat intervals and Kazhdan–Lusztig polynomials

Ewan Delanoy

Institut Camille Jordan, UMR 5028 CNRS, Université Lyon 1, 69622 Villeurbanne Cedex, France

Received 20 January 2006; accepted 11 January 2007
Available online 21 June 2007

Abstract

Our main result is that the recently proved combinatorial invariance property for Kazhdan–Lusztig polynomials on lower Bruhat intervals still holds for Bruhat intervals whose top element is critical. We conjecture that our theory also extends to completely compressible Bruhat intervals in type A, D, E, and we have checked this conjecture up to types A_8, D_7, E_6 with a computer. Positive results in related special cases are also presented.

1. Introduction and statement of the conjecture

Let (W, S) be a Coxeter system, and denote by $\leq$ the Bruhat ordering on W. The question asked independently by Dyer [10] and Lusztig, of whether the Kazhdan–Lusztig polynomial $P_{u,v}$ defined on a Bruhat interval only depends on the isomorphism class of $[u, v]$, remains open today (see [11] for all the definitions concerning the Bruhat ordering and P- and R-polynomials). Denote by $\mathcal{R}$ the set of all the isomorphisms that preserve R-polynomials (i.e. poset isomorphism $\phi : I \to I'$ such that $R_{\phi(u),\phi(v)} = R_{u,v}$ for any $u, v \in I$, where I and I' are Bruhat intervals); for brevity we call them R-isomorphisms. It has been shown recently (in [2] and also in [4]) that if $[e, y]$ and $[e, y']$ are two Bruhat intervals originating at the identity in their respective Coxeter groups, then

$$\text{Any isomorphism } \phi : [e, y] \to [e, y'] \quad \text{is in } \mathcal{R}. \tag{1.1}$$

E-mail address: delanoy@math.univ-lyon1.fr.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2007.01.006

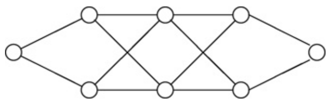

Fig. 1. The dihedral poset in length 4.

There are certainly several ways in which this result is likely to be extended; we discuss one of them here. The famous "lifting property" relating (left, say) multiplication by a generator to the Bruhat order has been the starting point for much work, starting with the "Z-property" introduced in [6] and generalized in ([10], section 5.15). More recently, it led Du Cloux [7] and Brenti, Caselli and Marietti [1–3] to develop independently very close theories. Thus, what is called a "compression" here and in [7] is called a "special matching" in [1–3]. Throughout this paper we reason wholly inside Du Cloux's framework, but in Appendix B we say a few more words about the connection between the two theories. A natural extension of the class of posets isomorphic to a Bruhat interval originating at the identity is the class of *completely compressible* posets, i.e. those that can be reduced to the trivial poset by a sequence of compressions (see Appendix B for the basics about compressions, and [7] for a more complete exposition). More precisely, consider C_1, the class of all completely compressible Bruhat intervals, and C_0, the subclass of C_1 consisting of the Bruhat intervals originating at the identity. Then maybe the following extension can be deduced from (1.1):

$$\text{Any isomorphism } \phi : P \to P' \text{ is in } \mathcal{R} \quad \text{if } P, P' \in C_1. \tag{1.2}$$

To prove (1.2), it would suffice to show that

$$\text{For any } P \in C_1, \text{ there is an } R\text{-isomorphism } \phi : P \to P_0, \quad \text{with } P_0 \in C_0. \tag{1.3}$$

Unfortunately, this is false in general; indeed, the interval $[3, 3(1212)3]$ in type B_3 (which means that $m_{12} = 4, m_{13} = 2, m_{23} = 3$) is completely compressible, but is not isomorphic to any $[e, y]$ (we explain the reasons for this in Appendix A). However, if we restrict our attention to simply laced finite Coxeter groups (i.e. to the A, D, E types), then (1.3) seems to hold; in fact, for this realm we make a much stronger conjecture, which implies that there is a uniquely defined isomorphism if we impose some additional conditions.

Let us develop some general tools about isomorphisms onto intervals originating at the identity. Recall that a Coxeter system is dihedral if its rank is 2, and that a poset is dihedral if it is isomorphic to an interval in a dihedral Coxeter group. For each length n, there is exactly one dihedral poset in length n, which has a minimum and maximum element, and two elements in each intermediate length (see Fig. 1).

In a poset P with minimum element x, an element $y \in P$ is said to be dihedral when the interval $[x, y]$ (as a subposet of P) is dihedral. The *bud* $B(P)$ of P is defined to be the set of the dihedral elements of P (cf. e.g. [9]). The following "$K_{3,2}$-avoidance" proposition along with its corollaries is fundamental:

Proposition 1.1 (*[2], Theorem 3.2*). *Let (W, S) be a Coxeter system. If two elements of W have three coatoms (or three atoms) in common, they are equal.*

From now on, $I = [x, y]$ will always denote a Bruhat interval.

Corollary 1.2. *The dihedral elements of I are exactly the elements of I that have at most two coatoms.*

Corollary 1.3. *The coatoms set function is injective on $I \setminus B(I)$.*

For any two atoms a, b of I define a subset $D^I(a, b)$ of $B(I)$ as follows:

$$D^I(a, b) = \{x; a; b\} \cup \{z \in B(I) | z \geq a, z \geq b\}.$$

If I is completely compressible, all the $D^I(a, b)$ are closed intervals (see Corollary B.6). Define $m^I(a, b)$ to be the length of the interval $D^I(a, b)$. If $\phi : I \to [e, w']$ is an isomorphism onto an interval originating at the identity (in a possibly different Coxeter system (W', S')), then restricting S' if necessary we may identify S' with the set $\mathsf{atoms}(I)$ of the atoms of I. Then, because $\phi(D^I(a, b))$ is dihedral in W', we must have $m'_{a,b} \geq m^I(a, b)$ for any two atoms a, b of I. In addition, we have the following elementary result from Proposition 3.5 of [9]:

Proposition 1.4. *Let $[e, w']$ be a lower interval in some Coxeter system (W', S'). Let $(W_{\text{can}}, S_{\text{can}})$ be a Coxeter system defined as follows: $S_{\text{can}} = \{s \in S'; s \leq w'\}$ (the support of w'), and for $s, t \in S_{\text{can}}$, $(m_{\text{can}})_{s,t}$ is the length of the closed dihedral interval $D^{[e,w']}(s, t)$, as above (in this special case $D^{[e,w']}(s, t)$ is the intersection of $[e, w']$ with the dihedral subgroup $\langle s, t\rangle$). Then there is a uniquely defined isomorphism $\zeta : [e, w'] \to [e, w_{\text{can}}]$ (with $w_{\text{can}} \in W_{\text{can}}$) satisfying*

$$\zeta(su) = s\zeta(u) \; (\zeta(us) = \zeta(u)s) \quad \textit{whenever } s \in S_{\text{can}}, u \in [e, w'], su \in [e, w'] \; (us \in [e, w'])$$

and ζ preserves R-polynomials.

This motivates the following definition:

Definition 1.5. Let I be a completely compressible Bruhat interval. Define a Coxeter system $\mathcal{S} = (W_{\text{can}}, S_{\text{can}})$ as follows: $S_{\text{can}} = \mathsf{atoms}(I)$ and $(m_{\text{can}})_{a,b} = m^I(a, b)$ for $a, b \in S'$. We call $\mathcal{S}$ the *canonical image Coxeter system* associated with I. A *standard isomorphism* ϕ of I is an isomorphism from I onto an interval originating at the identity in W_{can}, such that $\phi(a) = a$ for any a in $\mathsf{atoms}(I)$; and we may paraphrase now the aforementioned result:

Proposition 1.6. *For any isomorphism $\phi : I \to [e, w']$ (where the image Coxeter system (W', S') may be arbitrary), there is another isomorphism $\zeta : [e, w'] \to [e, w'']$ such that $\zeta \circ \phi$ is standard.*

In other words, any isomorphism onto a $[e, w]$ may be standardized, so that we need only look for standard isomorphisms. The next proposition (which follows easily from 1.2 and 1.3) tells us just how much choice we have in the definition of a standard isomorphism:

Proposition 1.7. *Any standard isomorphism of I is uniquely defined by its restriction to $B(I)$ (which yields an isomorphism between the buds $B(I)$ and $B(W_{\text{can}})$).*

For example, if a, b are two generators in W with $m_{a,b} \geq 3$, τ is the transposition which exchanges ab and ba, then τ and id (the identity map) yield two distinct standard isomorphisms of $[e, ab]$. We now introduce some additional conditions on isomorphisms that will enable us to reject τ and accept id.

If $[x, y]$ is a Bruhat interval put $L_{[x,y]} = \{s \in S; sx \in [x, y]\}$ and $R_{[x,y]} = \{s \in S; xs \in [x, y]\}$.

Definition 1.8. An isomorphism ϕ between two Bruhat intervals $I = [x, y]$ and I' (an interval originating at the identity in a possibly different Coxeter group W') is *left-regular* if there is a (necessarily unique) injection $\lambda : L_{[x,y]} \to S'$ such that $\phi(sw) = \lambda(s)\phi(w)$ holds for any $s \in L_{[x,y]}$ and $w \in [x, y]$ such that $sw \in [x, y]$. It is *right-regular* if there is an injection $\rho : R_{[x,y]} \to S'$ such that $\phi(ws) = \phi(w)\rho(s)$ holds for any $s \in R_{[x,y]}$ and $w \in [x, y]$ such that $ws \in [x, y]$. It is *biregular* if it is both left-regular and right-regular.

Note that λ (or ρ) is "necessarily unique" because we must have $\phi(sx) = \lambda(s)\phi(x) = \lambda(s)$ for any $s \in L_{[x,y]}$. At this point, we naturally ask the question: Does the biregularity requirement ensure uniqueness for a standard isomorphism? The answer is no. We provide a counterexample in type A_6 (which is minimal in the sense that no counterexample exists in type A_5, and no example with an x of smaller length exists in type A_6).

Consider the dihedral interval

$$I = [w_L w_R, w_L 343 w_R] = \{w_L w_R, w_L 3 w_R, w_L 4 w_R, w_L 34 w_R, w_L 43 w_R, w_L 343 w_R\}$$

where $w_L = 215$, $w_R = 265$. The atoms of I (which are the same thing as the atoms of $x = w_L w_R$ in I) are $a = w_L 3 w_R$ and $b = w_L 4 w_R$, so that there are reflections l_a, r_a, l_b, r_b such that $a = l_a x = x r_a$, $b = l_b x = x r_b$ (and for example $l_a x$ is reduced exactly when l_a is a generator). A little computation shows that

$$l_a = r_a = 232$$
$$l_b = r_b = 454.$$

None of those reflections is a generator; we deduce $L_I = R_I = \emptyset$, so that any standard isomorphism on I is trivially biregular. It is easily seen that there are exactly two standard isomorphisms of I (as in the diagram below)

w	$w_L w_R$	$w_L 3 w_R$	$w_L 4 w_R$	$w_L 34 w_R$	$w_L 43 w_R$	$w_L 343 w_R$
$\frac{\phi_1(w)}{\phi_2(w)}$	e	a	b	$\frac{ab}{ba}$	$\frac{ba}{ab}$	$aba = bab$

In this special case, we feel inclined to accept ϕ_1 and reject ϕ_2 because "3 corresponds to a and 4 corresponds to b", but it seems wildly improbable that such a naive approach will be successful with more complicated counterexamples. Surprisingly, this idea can be nicely formalized and provide a conjecture that has withstood the test of reasonably large examples by computer.

Until now, we have combined left and right actions of the generators (as in the definition of biregularity). From now on, we shall always act from the left only; the primary reason for this restriction is to circumvent the complex interaction between left and right (this will become clear with Proposition 1.14, which holds only for a one-sided action). Some propositions like 1.23 are also valid in a two-sided context, but here we present only the 'left' version for the sake of simplicity.

We now proceed to the formal definitions. The following proposition is well known (see Proposition B.4(i) and Proposition 2.7 of [7]):

Proposition 1.9. *Let $I = [x, y]$ be a Bruhat interval, and let $s \in S$ be such that $x > sx$, $y > sy$. Then the following are equivalent:*

(i) $x \not\leq sy$.
(ii) *Any $w \in [x, y]$ satisfies $w > sw$.*
(iii) *Left multiplication by s is a poset isomorphism $[x, y] \to [sx, sy]$.*

Definition 1.10. Let $I = [x, y]$, s be as in 1.9 above. If (i), (ii) or (iii) is satisfied we say that $[sx, sy]$ is a *(left) elementary lower translate* of $[x, y]$. We say that an interval J is a *(left) lower translate* of I if there is a chain of intervals $I_0 = I, I_1, \dots I_n = J$ such that I_{i+1} is a (left) elementary lower translate of I_i; or equivalently if there is an element t_L in W and an isomorphism $\theta : J \to I$, which satisfies $\theta(w) = t_L w, l(\theta(w)) = l(t_L) + l(w)$ for all $w \in I$. We call such an isomorphism a *(left) translation*. If $\phi : I \to I'$ is any isomorphism onto an interval I' into a possibly different Coxeter group, then we have a new isomorphism $\psi = \phi \circ \theta : J \to I'$. We call ψ a *(left) lower translate* of ϕ. If $c : I \to I$ is any compression of I, then we have a compression of J: $c' = \theta^{-1} \circ c \circ \theta : J \to J$. We call c' a *(left) lower translate* of c.

Clearly, left translations preserve the R-polynomials. Note that the left lower translate of a left-regular isomorphism is not necessarily left-regular. Indeed, take W to be the free Coxeter group on the generators l, s_1, s_2, and consider the intervals

$$I = [l; ls_1s_2] = \{l; ls_1; ls_2; ls_1s_2\}$$
$$J = [e; s_1s_2] = \{e; s_1; s_2; s_1s_2\}$$
$$I' = [e; s_2s_1] = \{e; s_1; s_2; s_2s_1\}.$$

The interval J is a lower translate of I, with left multiplication by l for a translation. Then the isomorphism $\phi : I \to I'$ defined by $\phi(l) = e, \phi(ls_i) = s_i, \phi(ls_1s_2) = s_2s_1$ is left-regular, but the associated ψ is not left-regular.

Definition 1.11. A left-regular isomorphism ϕ is *strongly left-regular* if all its left lower translates are left-regular.

Note that if ϕ is a strongly left-regular isomorphism, then a lower restriction of ϕ (i.e. the restriction of ϕ to a lower interval of $[x, y]$) is not necessarily strongly left-regular. To see this, let ϕ be the non-strongly left-regular example isomorphism on $I = [l, ls_1s_2]$ defined above. Add a new generator l', and consider the free Coxeter group on l, l', s_1, s_2. Extend ϕ to a mapping ϕ' on the larger interval $I_1 = [l, l'ls_1s_2]$ by putting $\phi'(l'w) = l'\phi(w)$ for $w \in I$. Then ϕ' is strongly left-regular (in fact I_1 has no left lower translate other than itself) but its lower restriction ϕ is not.

Definition 1.12. A strongly left-regular isomorphism ϕ is *totally left-regular* if all its left lower restrictions are strongly left-regular.

Trivially, we have:

Remark 1.13. If ϕ is totally left-regular, then so are its left lower translates and lower restrictions.

Next we show a lattice property:

Proposition 1.14. *Let I be a Bruhat interval and let J, J' be two left lower translates of I. Then there is an interval K which is a left lower translate of both J and J'.*

Note that the 'two-sided' version of this does not hold in general: take $I = [12, 121]$, $J = [e, 1]$, $J' = [e, 2]$ in type A_2. Then J is a left lower translate of I and J' is a right lower translate of I, but neither J nor J' can be translated further.

Proof. We may assume that J and J' are elementary lower translates of I, the general case following by induction. Clearly, we may also assume that $J \neq J'$. Thus $J = sI$, $J' = s'I$ with

$s, s' \in S, s \neq s'$. Then $s \neq s'$ because $J \neq J'$; for any $w \in I$ both s and s' are in the left descent set of w, so $m = m_{s,s'}$ is finite, and putting $b = ss'ss' \dots (m \text{ terms}) = s'ss's \dots (m \text{ terms})$ we can write $w = bw', l(w) = l(b) + l(w')$ for some $w' \in W$. Then we can take $K = (b^{-1})I$. □

Corollary 1.15. *If I is a Bruhat interval there is a lowest left lower translate of I (we call it the left core of I and denote it by $\mathsf{core}(I)$). As in Definition 1.10 one can likewise define the left core of an isomorphism defined on I or the left core of a compression of I.*

As explained above, there is no notion of a 'two-sided core': in general there are several minimal lower translates for a given interval I (e.g. $[e, 1]$ and $[e, 2]$ for $[12, 121]$). In the cases where our final conjecture holds, however, uniqueness ensures that the 'left' construction yields the same object as the 'right' construction, so that the propositions below hold in a two-sided context, with 'any minimal lower translate' in place of 'left core'.

The following lemma is fundamental:

Lemma 1.16. *Let $I = [x, y]$ be a Bruhat interval, $J = \mathsf{core}(I)$ and θ the associated translation isomorphism $J \to I$. Let s be a generator such that $x < sx < sy < y$. Then there is a generator s' such that $\theta^{-1}(sx) = s'\theta^{-1}(x)$ and $\theta^{-1}(sy) = s'\theta^{-1}(y)$.*

Proof. We argue by induction on $l(y)$. If $l(y) = 0$ there is nothing to prove, so suppose $l(y) > 0$. Obviously we may assume that $J \neq I$. Since we reason by induction, it is not necessary to show the property on J at once; it suffices to show it on an intermediary left lower translate: in other words, it suffices to find a left lower translate K of I with $K \neq I$ such that for the associated translation isomorphism $\kappa : K \to I$, there is a generator s' such that $\kappa^{-1}(sx) = s'\kappa^{-1}(x)$ and $\kappa^{-1}(sy) = s'\kappa^{-1}(y)$.

There is an element t_L in W such that $\theta(w) = t_L w, l(\theta(w)) = l(t_L) + l(w)$ for all $w \in J$. Then $t_L \neq e$ since $J \neq I$; let t be a generator in the left descent set of t_L; clearly $t \neq s$. Then both s and t are in the left descent set of y, so $m = m_{s,t}$ is finite, and putting $b = stst \dots (m \text{ terms}) = tsts \dots (m \text{ terms})$ we can write $y = by', l(y) = l(b) + l(y')$ for some $y' \in W$. Similarly, there is an element $x' \in W$ such that $sx = bx', l(sx) = l(b) + l(x')$.

Consider the interval $L = t[x, y] = [tx, ty] = [\underbrace{(st\dots)}_{m-2 \text{ terms}} x', \underbrace{(st\dots)}_{m-1 \text{ terms}} y']$. Then L is a lower translate of I; in particular, no element of L has t in its left descent set. We deduce that if $m > 2$, $\underbrace{(ts\dots)}_{m-2 \text{ terms}} y' \notin L$. Thus, $L' = sL$ is a lower translate of L (this is condition (i) of 1.10). Similarly, if $m > 3$ then $L'' = tL'$ is a lower translate of L'. Continuing in this way, we eventually obtain a translation decomposition $L = \underbrace{(st\dots)}_{m-2 \text{ terms}} [x', cy']$ where c is the last generator in $\underbrace{(st\dots)}_{m-1 \text{ terms}}$ (thus $c = t$ if m is odd and $c = s$ otherwise). Then $K = [x', cy']$ is a lower translate of I, and taking $s' = c$ we are done. □

Here is a typical application of this lemma:

Proposition 1.17. *Let ϕ be an isomorphism from I onto an interval originating at the identity. Then ϕ is totally left-regular if and only if $\mathsf{core}(\phi)$ is.*

Proof. Only the "if" part deserves attention, of course. So suppose that $\mathsf{core}(\phi)$ is totally left-regular. We must show that ϕ is totally left-regular, i.e. that for any lower interval I' of I, the restriction of ϕ to I' is strongly left-regular. But if θ is the translation isomorphism $\mathsf{core}(I) \to I$,

$$\begin{tikzcd}
{[x,y]} \arrow[d] & \theta^{-1}[x,y] \arrow[l] \arrow[d] \arrow[dr, "\psi"] & \mathsf{core}([x,y]) \arrow[l] \arrow[d, "\psi'"] \\
I \arrow[rr, bend right, "\phi"'] & \mathsf{core}(I) \arrow[l] \arrow[r, "\mathsf{core}(\phi)"'] & \text{Image of } \phi
\end{tikzcd}$$

Labelled arrows : injective poset homorphisms
For the other arrows :
Horizontal arrows : translations
Vertical arrows : inclusions

Fig. 2. Relating an isomorphism to its core.

then $\theta^{-1}(I')$ is a left lower translate of I', so $\mathsf{core}(I')$ is a left lower translate of $\theta^{-1}(I')$. Thus we may replace I' with I, and all we need to show is that ϕ is strongly left-regular. For any left lower translate K of I we have $\mathsf{core}(K) = \mathsf{core}(I)$ so we may replace K with I, and all we need to show is that ϕ is left-regular.

Let us show that ϕ is left-regular. Suppose that we have $x < sx < sy < y$ where x is the smallest element of I and $y \in I$. Let θ be the translation isomorphism $\mathsf{core}(I) \to I$; then $\mathsf{core}([x, y])$ is a lower translate of $\theta^{-1}([x, y])$, whence a translation isomorphism $\alpha : \mathsf{core}([x, y]) \to \theta^{-1}([x, y])$. By the lemma above there is a generator s' such that $(\theta\alpha)^{-1}(sx) = s'(\theta\alpha)^{-1}(x)$ and $(\theta\alpha)^{-1}(sy) = s'(\theta\alpha)^{-1}(y)$. Let ψ be the restriction of $\mathsf{core}(\phi)$ to $\theta^{-1}([x, y])$. Then ψ is totally left-regular, and so is its left lower translate $\psi' = \psi \circ \alpha$. In particular, if we write $\mathsf{core}([x, y]) = [x', y']$, we have $\psi'(sy') = \psi'(sx')\psi'(y')$. In terms of ϕ, this means that $\phi(sy) = \phi(sx)\phi(y)$ as required. The situation is depicted graphically in Fig. 2 above. □

Definition 1.18. A compression c of a Bruhat interval I is *explicit* if some lower translate of it is a (left or right) multiplication by a generator.

Thus if c is a explicit compression of I there is a translation $\theta : J \to I$ and an $s \in S$ such that $c(w) = \theta(s\theta^{-1}(w))$ (say). But θ can be defined by $\theta(w) = t_L w t_R$ for some fixed $t_L, t_R \in W$ and all $w \in J$. Then $c(w) = (t_L s t_L{}^{-1})w$ for any $w \in W$, an explicit formula which justifies the term. One may wonder about the converse:

Question 1.19. *Is it true that if c is an explicit compression of I such that there is a $r \in W$ (which will necessarily be a reflection) such that $c(w) = rw$, then c is a left explicit compression?*

Definition 1.20. Let I and J be two Bruhat intervals. We say that J is a *(left) multiplication compression of I* if we can write $I = [x, y]$, $J = [x, sy]$ for some $x, y \in W$ and some $s \in S$ such that $x < sx, sy < y$.

The following notion is fundamental:

Definition 1.21. Let $I = [x, y]$ be a Bruhat interval. We say that I is *left explicitly completely compressible* (*l.e.c.c.* in abbreviated notation) if

(*) there is a chain of intervals $I_0 = I, I_1, \dots I_n = [z, z]$ such that I_{i+1} is either a left elementary lower translate of I_i or a left multiplication compression of I_i.

Note that if s_1 and s_2 are two generators with $m(s_1, s_2) \geq 4$ and $I = [s_1, s_1 s_2 s_1]$ then I has exactly two compressions and both are non-explicit. Thus, I is not explicitly completely compressible.

The following two propositions show the usefulness of the notion of (left) explicit complete compressibility:

Proposition 1.22. *If I is l.e.c.c., then there is a unique totally left-regular standard isomorphism ϕ on I.*

Proposition 1.23. *Let $\phi : I \to [e, y]$ be an isomorphism, with ϕ totally left-regular and I l.e.c.c. Then ϕ preserves R-polynomials.*

The proof of the second proposition is simpler and we begin with this one.

Proof of Proposition 1.23. Call $l^*(I)$ the smallest n such that there exists $I_0 = I, I_1, I_2, \ldots, I_n = [z, z]$ as in 1.21(*). We argue by induction on $l^*(I)$. If $l^*(I) = 0$ there is nothing to prove, so suppose $l^*(I) > 0$. There is a sequence $I_0, \ldots I_n$ as in 1.21(*). Then $J = I_1$ satisfies $l^*(J) = l^*(I) - 1$ and J is either an elementary left lower translate of I or a left elementary compression of I. In the first case, the lower translate isomorphism $\psi = \phi \circ \theta : J \to [e, y]$ is in $\mathcal{R}$, and $\theta \in \mathcal{R}$ also, so $\phi = \psi \circ \theta^{-1} \in \mathcal{R}$. In the second case, we can write $J = [x, sy]$, $I = [x, y]$ as in 1.20. By the induction hypothesis, the restriction $\phi_{|J}$ of ϕ to J is in $\mathcal{R}$. Let $u \le v$ in I. If $v \in J$, then $R_{u,v} = R_{\phi(u),\phi(v)}$ by the preceding remark. Otherwise $v' = sv < v$ and $v' \in J$. We have $\phi(v) = \lambda(s)\phi(v')$ by left-regularity, and $\phi(v') < \phi(v)$ since ϕ is an isomorphism. Then, if $su > u$, we have

$$\begin{aligned} R_{u,v} &= R_{u,sv'} \\ &= qR_{u,v'} + (q-1)R_{su,v'} \\ &= qR_{\phi(u),\phi(v')} + (q-1)R_{\phi(su),\phi(v')} \quad (\text{ because } v' \in J) \\ &= qR_{\phi(u),\phi(v')} + (q-1)R_{\lambda(s)\phi(u),\phi(v')} \\ &= R_{\phi(u),\lambda(s)\phi(v')} \\ &= R_{\phi(u),\phi(v)}. \end{aligned}$$

If $su < u$, we have a similar and simpler computation, so that in all cases, ϕ preserves R-polynomials as claimed. □

Proof of Proposition 1.22. Uniqueness is clear: the 'standard' condition defines the image Coxeter group completely, and any sequence sending I to a trivial interval as in 1.21(*) yields an explicit formula for $\phi(w)$, for each $w \in I$. Also, as standardizing an isomorphism (as explained in 1.4 and 1.6) does not affect the (simple, strong or total) left-regularity of it, all we need to show is the existence of a totally left-regular isomorphism of I. This will follow immediately from the next two lemmas: □

Lemma 1.24. *Let I be a Bruhat interval and let J be a left lower translate of I, so there is a translation isomorphism $\theta : J \to I$. If ϕ is a totally left-regular isomorphism of J, then $\phi \circ \theta$ is again a totally left-regular isomorphism of I.*

Lemma 1.25. *Let I be a Bruhat interval and let J be a left multiplication compression of I. If ϕ is a totally left-regular isomorphism of J, then ϕ may be extended to a totally left-regular isomorphism of I.*

Proof of Lemma 1.24. Since $\mathsf{core}(I) = \mathsf{core}(J)$, this is clear by Proposition 1.17. □

Proof of Lemma 1.25. There is a generator s such that $J = [x, y]$, $I = [x, sy]$ with $x < sx$, $y < sy$. There are two main cases, according to whether sx is in $[x, y]$ or not. In the first case, the generator s is "already known" inside J and in the second, we have to add a new generator to our image Coxeter system. We only explain the first case here, because the second is similar and easier.

So suppose $sx \in [x, y]$; then the left mapping λ associated with the left-regular mapping ϕ (as in 1.8) is defined at s. We (naturally) extend ϕ to a mapping defined on $[x, sy]$ as follows:

$$\text{For } w \in I, \phi'(w) = \begin{cases} \phi(w) & \text{if } w \leq y \\ \lambda(s)\phi(sw) & \text{if } w \not\leq y. \end{cases}$$

Since ϕ is a left-regular isomorphism, we have for all $u \in J$ that s is in the left descent set of u if and only if $\lambda(s)$ is in the left descent set of $\phi(u)$. We deduce that ϕ' is an isomorphism between two Bruhat intervals.

By Proposition 1.17, it suffices to show that $\mathsf{core}(\phi')$ is totally left-regular. By Lemma 1.16, there is a generator s' such that if θ is the translation isomorphism $I \to \mathsf{core}(I)$, then $\theta(sx) = s'\theta(x')$ and $\theta(y) = s'\theta(sy)$. Thus we may replace I with $\mathsf{core}(I)$; in other words, we may assume $\mathsf{core}(I) = I$.

We must show that ϕ' is totally left-regular, i.e. that for any lower interval $[x, z]$ of I, the restriction of ϕ' to $[x, z]$ is strongly left-regular. If $z \leq y$ this follows from the total left-regularity of ϕ; otherwise z is of the form $sz' > z'$ with $z' \leq y$, and replacing (z', z) with (y, sy), all we need to show is that ϕ' is strongly left-regular. But since we have assumed $\mathsf{core}(I) = I$, left-regularity is equivalent to strong left-regularity for ϕ'. So all we need to show is that ϕ' is left-regular.

Left-regularity with respect to s is obvious from the construction. Let us show left-regularity with respect to a generator $t \neq s$. So suppose $x < tx < tz < z$ for some $z \in I$; we must show $\phi'(tz) = \phi'(tx)\phi'(z)$. We claim that $tx \leq y$. Otherwise we could write $tx = sx'$ for some $x' \leq y$ with $x' < sx'$; in particular s would be in the left descent set of tx, and of x also, since $t \neq s$, so $x > sx$ which is absurd. So $tx \in [x, y]$; we deduce that ϕ is left-regular with respect to both s and t, and the result follows. □

Definition 1.26. Let $I = [x, y]$ be a Bruhat interval and let s be a generator in the (left) descent set of y (we then say that s is a *(left) descent generator* for I). If $x < sx$ (so that $[x, y]$ can be compressed onto $[x, sy]$) we say that s is a *(left) compression generator* for I. If $sx < x$ and $x \not\leq sy$ (so that there is a translation isomorphism $[sx, sy] \to [x, y]$), we say that s is a *(left) translation generator* for I. Otherwise $sx < x$ and $x < sy$; in this last case we say that s is a *(left) nontrivial descent generator* for I.

Conjecture 1.27. *If I is a completely compressible Bruhat interval in type A, D or E, then I has at least one trivial left descent generator.*

Note that this is equivalent to the statement that any completely compressible Bruhat interval in type A, D or E is l.e.c.c. We have checked Conjecture 1.27 up to types A_8, D_7, E_6 with a specialized version of the program `Coxeter` [8].

Using Propositions 1.22 and 1.23, Conjecture 1.27 implies that (1.3) (and hence (1.2)) holds in types A, D, E.

2. Special cases of the conjecture

Recall that for any interval I, $\mathsf{coat}(I)$ denotes the set of coatoms of I, and for $w \in W$, $\mathsf{coat}([e, w])$ is abbreviated as $\mathsf{coat}(w)$. The inequality below is obtained by a straightforward induction on $l(I)$:

Remark 2.1. If I is completely compressible, we have $|\mathsf{coat}(I)| \leq l(I)$. In particular, for $w \in W$ we have $|\mathsf{coat}(w)| \leq l(w)$.

Definition 2.2. Let W be a Coxeter group and $w \in W$. We say that $w \in W$ is *critical* if $|\mathsf{coat}(w)| = l(w)$.

Note that critical elements are fully commutative (see [12] for more on fully commutative elements of Coxeter groups). The converse is already false in a dihedral Coxeter group of length ≥ 4, and it is also false in type $\tilde{A}_2$ (taking generators s_1, s_2, s_3 with $m(s_i, s_j) = 3$ for $i \neq j$, the element $w = s_1 s_2 s_3 s_1 s_2$ is fully commutative but not critical). However, we do not know the answer to the following question:

Question 2.3. *If W is of type A, D or E, is it true that the critical elements of W are exactly the fully commutative elements of W?*

Our main result is:

Theorem 2.4. *Let $I = [x, y]$ be a Bruhat interval with y critical. If I has at least a nontrivial descent generator, then $|\mathsf{coat}(I)| > l(I)$ so that I is not completely compressible by Remark 2.1.*

Corollary 2.5. *If $I = [x, y]$ is a completely compressible Bruhat interval with y critical, then all the descent generators of I are trivial.*

Proof of the theorem. Let s be a (left, say) nontrivial descent generator for I. Then we can write $x = sx'$, $y = sy'$ with $x' < x$, $y' < y$. Since s is nontrivial we have $x < y'$. Let $a = y_1 \dots y_n$ be a reduced expression of y'. We know there are r indices $i_1 < i_2 < \cdots < i_r$ (with $r = l(y) - l(x)$) such that if we delete the generators $y_{i_1}, \dots, y_{i_r}$ from a, we obtain a reduced expression for x'. Now put $c_k = s y_1 \dots \widehat{y_{i_k}} \dots y_n$ for k between 1 and r. Since y is critical, c_k is a coatom of y. Then $c_1, c_2, \dots c_r, y'$ are all coatoms of I, which provides at least $r + 1$ coatoms as desired. $\square$

Corollary 2.6. *If $[x, y]$ is a completely compressible interval with y critical, then it is left explicitly completely compressible.*

Indeed, if $s_1 s_2 \dots s_m$ is a reduced expression for y, then there is a sequence of intervals as in 1.21(ii), $I_0 = [x, y]$, $I_1 = [x_1, y_1] \dots I_n = [x_n, y_n]$ with $x_n = y_n$, defined as follows:

$$x_i = \min(x_{i-1}, s_i x_{i-1})$$
$$y_i = s_i y_{i-1} \quad \text{(so that } y_i = s_{i+1} \dots s_m).$$

This sequence stops at the first n for which $x_n = y_n$.

Proposition 2.7. *Suppose that W is of type A_n. If $w \in W$ is a Grassmannian (i.e. $|D_l(w)| \leq 1$), then w is critical.*

Proof. We use the well-known representation of W as the symmetric group on $\{1, 2, \dots, n + 1\}$, where the i-th generator is represented by the transposition $s_i = (i, i + 1)$. Then one has for $i < j$ that $l(w(i, j)) < l(w)$ if and only if $w(i) > w(j)$ and $l((i, j)w) < l(w)$ if and only if $w^{-1}(i) > w^{-1}(j)$. Put

$$T_w = \{(i, j) | i < j, l((i, j)w) < l(w)\}.$$

Then it is well known that $|T_w| = l(w)$, and we have $D_l(w) = T_w \cap S$, $D_r(w) = D_l(w^{-1})$.

Since $[e, w] \approx [e, w^{-1}]$ and $|D_l(w)| = 0$ implies that $w = e$, it will suffice to show that if $|D_r(w)| = 1$ then w is critical. So assume that $D_r(w) = (l, l+1)$ for some $l \in [1..n]$; we must show that $l(tw) = l(w) - 1$ for all $t \in T_w$. Define $m = (n+1) - l$, $a_i = w(i)$ for $1 \leq i \leq l$ and $b_j = w(l+j)$ for $1 \leq j \leq m$. Then $a_1 < a_2 < \cdots < a_l$, $b_1 < b_2 < \cdots < b_m$ and $a_l > b_1$. The set T_w consists exactly of the transpositions (b_j, a_i) such that $b_j < a_i$. Let $t \in T_w$. Then the permutations w and tw differ only by their values at i and $l + j$. Therefore,

$$\{(p, q) \in T_w | \{p, q\} \cap \{a_i, b_j\} = \emptyset\} = \{(p, q) \in T_{tw} | \{p, q\} \cap \{a_i, b_j\} = \emptyset\}$$

so we need only show $|A| = |B| + 1$ where

$$A = \{(p, q) \in T_w | \{p, q\} \cap \{a_i, b_j\} = \emptyset\}, \qquad B = \{(p, q) \in T_{tw} | \{p, q\} \cap \{a_i, b_j\} = \emptyset\}.$$

Choose k maximal with $a_i > b_k$ and p minimal with $b_j < a_p$. Then $|\{a | (b_j, a) \in T_w\}| = l - p + 1$ and $|\{b | (b, a_i) \in T_w\}| = k$, so by counting (b_j, a_i) twice we deduce $|A| = (l - p + 1) + k - 1 = l - p + k$. On the other hand, $|\{a | (b_j, a) \in T_w\}| = (i - p) + (j - 1)$ and $|\{b | (b, a_i) \in T_w\}| = (l - i) + (k - j)$ so $|B| = (i - p) + (j - 1) + (l - i) + (k - j) = l + k - p - 1$, i.e. $|B| = |A| - 1$ as required. □

We deduce:

Corollary 2.8. *Let $I = [x, y]$ be a completely compressible Bruhat interval in a Coxeter group W. Then I has a trivial left descent generator in any of those three cases:*

-(i) $D_l(y) \not\subseteq D_l(x)$.
-(ii) *W is of type A and y is a Grassmannian.*
-(iii) *W is of type A and x is a Grassmannian.*

Note that in (i) any generator in $D_l(y) \setminus D_l(x)$ is a compression generator for I, and that (iii) follows from (i) and (ii).

To conclude, let us review to what extent we answered our original question:

Proposition 2.9. *Consider the following classes of Bruhat intervals:*

- $\mathcal{F}_1 = \{[x, y] \mid [x, y]$ *is completely compressible,* y *is critical*$\}$.
- $\mathcal{F}_2 = \{$*Completely compressible intervals in type* A_8, D_7 *or* $E_6\}$.
- $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

Then for each element I in $\mathcal{F}$ there is a (standard totally left-regular) isomorphism from I onto an interval originating at the identity. Any isomorphism between two elements of $\mathcal{F}$ preserves R-polynomials (and hence Kazhdan–Lusztig polynomials also).

Perhaps the ideas of this paper could be improved to provide a result that covers all completely compressible intervals. A better understanding of the counterexample in type B_3 might be a starting point.

Acknowledgements

The author would like to express his thanks to both the anonymous referees, who provided many corrections and suggestions. In particular, Proposition 2.7 of this paper is due to one of the referees. Most of the material of this paper comes from the author's doctoral thesis [5].

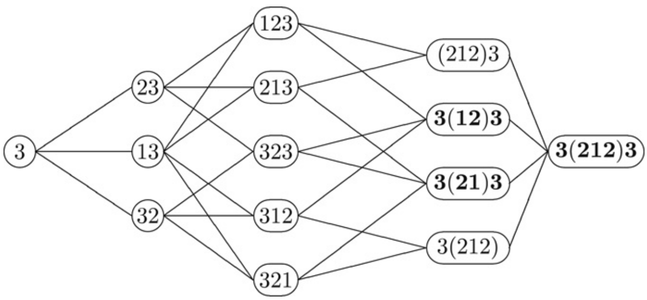

Fig. 3. The Bruhat interval [3, 32123] in type B3.

Appendix A. Explanations on the example in type B_3

The interval $I = [3, 32123]$ consists of fourteen elements, as described in Fig. 3 above (non-dihedral elements in boldface):

It is not too difficult to see that I is isomorphic to the lower interval $I' = [e, abca]$ in type A_3 (that is, $m_{a,b} = m_{b,c} = 3, m_{a,c} = 2$). So I is completely compressible. Now, if we put $J = [3, 3(1212)3]$, then left multiplication by 1 compresses J into I; thus J is completely compressible.

Suppose that there is an isomorphism $\phi : J \to [e, y]$ where $[e, y]$ is an interval originating at the identity in some Coxeter system (W', S'). Then $a = \phi(23), b = \phi(13), c = \phi(32)$ are elements of S'. We may assume that the isomorphism is standard, i.e. that $m_{ab} = m_{bc} = 4$, $m_{ac} = 2$. Then $\phi((1212)3) = abab, \phi(3(1212)) = bcbc, \phi(232) = ac$.

Let $u \in \{121; 212\}$. We have $\phi(u3) \in \{aba; bab\}$ and symmetrically $\phi(3u) \in \{bcb; cbc\}$ whence we deduce $\phi(3u3) \in \{bacb; bcab\}$, so $\phi(u3) = bab, \phi(3u) = bcb$, but this must be true for two distinct values of u, which contradicts the fact that ϕ is one-to-one. □

Appendix B. Elementary results about compressions

Nothing in this appendix is new. We discuss some results and connections arising naturally in the context of compressions. Throughout this appendix, P is an eulerian poset with lowest element x and largest element y. We refer the reader to [7,3] for a more complete treatment (in particular, [3] contains definitions for non-eulerian posets). Following [7], we define:

Definition B.1. Let $\tau : P \to \{-, +\}$ be a labelling of the elements of P. We say that τ is a *compression labelling* if:

(a) One has $\tau(x) = +$ and $\tau(y) = -$.
(b) For each $u \in P$ that is labelled $+$, there is a unique $v = c_\tau(u) \in P$ covering u such that $\tau(v) = -$.
(c) For each $v \in P$ that is labelled $-$, there is a unique $u = c_\tau(v) \in P$ covered by v such that $\tau(u) = +$.

This defines an involution c_τ on P, called a *compression*.

Proposition B.2. *Let $c : P \to P$ be a matching of the Hasse diagram of P (i.e. c is involutive and either cu covers u or u covers cu for any $u \in P$). Then the following are equivalent:*

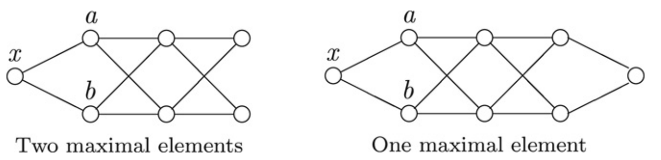

Fig. 4. The two cases for $D^P(a, b)$.

(i) *There exists a compression labelling* $\tau : P \to \{-, +\}$ *such that* $c = c_\tau$ *(so that c is a compression in the sense of* [7]*).*
(ii) *If* $u, v \in P$ *and* $u \lhd v, v \neq c(u)$*, then* $c(u) \leq c(v)$ *(so that c is a special matching in the sense of* [1]*).*
(iii) *If* $u \in P$ *and* $u \lhd c(u)$*, then* $\mathsf{coat}(c(u)) = \{u\} \cup \{c(z) | z \lhd u, z \lhd c(z)\}$.

Proof. (i) $\Rightarrow$ (ii). Suppose that (i) holds for some $\tau : P \to \{-, +\}$. Let $u, v \in P$ be such that $u \lhd v, v \neq c(u)$. Suppose that $\tau(u) = +$. Then $\tau(v) = -$ by B.1(b). Since P is eulerian, the open interval $]u, c(v)[$ consists of two elements, one of which is v. Call the other one x. Then $\tau(x) = -$ by B.1(a), and $x = c(u)$ by B.1(b). Therefore $c(u) \lhd c(v)$. The argument is similar in the two remaining cases ($\tau(u) = -$, and $\tau(v) = +$ or $-$).

(ii) $\Rightarrow$ (iii). Put $Z = \{u\} \cup \{c(z) | z \lhd u, z \lhd c(z)\}$. Let $w \in \mathsf{coat}(c(u))$. If $w = u$, we certainly have $w \in Z$. If $w \neq u$, using (ii) with w, cu in place of u, v we see that the element $z = cw$ is covered by u, so $w \in Z$. Therefore $\mathsf{coat}(c(u)) \subseteq Z$. The converse is likewise easy.

(iii) $\Rightarrow$ (i). Define $\tau : P \to \{-, +\}$ by $\tau(u) = +$ if $u \lhd c(u)$ and $\tau(u) = -$ if $c(u) \lhd u$. Then τ is a compression labelling. $\square$

If P is eulerian and c is a compression of P, and $x(y)$ is the lowest (largest) element of P, the mapping c sends P to the lower interval $[x, c(y)]$. We say that c *compresses* P onto $[x, c(y)]$.

Definition B.3. We say that P is *completely compressible* if there is a sequence $P_0, P_1, \ldots P_n = P$ such that P_0 is the trivial poset, and for each $i > 0$ there is a compression c_i of P_i that compresses P_i onto P_{i-1}.

Using Proposition 2.7, Lemma 2.13, and Theorem 2.17 of [7], respectively, we see that

Proposition B.4. *Let c be a compression of* P*. Then:*

(i) *If* $sx \not\leq sy$*, then c induces a poset isomorphism* $[sx, y] \to [x, sy]$*, and* P *is isomorphic to the graded product* $\{0, 1\} \times [sx, y]$.
(ii) *If* $u, v \in P$ *satisfy* $u \leq v, u < c(u), c(v) < v$*, then the interval* $[u, v]$ *is invariant by c.*
(iii) *If* P *is completely compressible, so are all its lower intervals.*

Note that (iii) implies that for an eulerian poset P, being completely compressible is the same thing as being a "zircon" in the sense of [3]. Recall the bud $B(P)$ of P and the sets $D^P(a, b)$ that we defined in the introduction:

$$B(P) = \{u \in P | [x, u] \text{ is a dihedral interval}\}$$
$$D^P(a, b) = \{x; a; b\} \cup \{z \in B(P) | z \geq a, z \geq b\}.$$

Assume that P is $K_{2,3}$-avoiding (i.e. if two elements of P have three atoms in common, they are equal). Then there are only two possibilities for $D^P(a, b)$, as shown in Fig. 4: either it has a maximum element and is a closed Bruhat interval, or it has two maximal elements.

Proposition B.5. *Suppose that P is $K_{2,3}$-avoiding and that there is a compression c that compresses P onto a lower interval P' of P. Let $a \neq b$ be two atoms of P. Then :*

(i) *If $c(x) \notin \{a, b\}$, then $D^P(a, b) \subseteq P'$ and hence $D^P(a, b) = D^{P'}(a, b)$.*
(ii) *If $c(x) \in \{a, b\}$, then $D^P(a, b)$ is a closed interval invariant by c.*

Proof. (i) Let $w \in D^P(a, b)$. Suppose by contradiction that $w \notin P'$. Then by B.4(ii), the interval $[x, w]$ is invariant by c, and hence $cx \leq w$. But then $w \geq a$, $w \geq b$, $w \geq c(x)$, contradicting the fact that w is dihedral.

(ii) Put $D = D^P(a, b)$. Let $x_1 \in P$ be uniquely defined by $\{a, b\} = \{c(x), x_1\}$. Then by B.2(iii) we have $\mathsf{coat}(c(x_1)) = \{a, b\}$ and hence $c(x_1) \in D$, so that $[x, c(x_1)] \subseteq D$. If $D = [x, c(x_1)]$ we are done. Otherwise, D contains an element x_2 of length 2 other than $c(x_1)$. By B.2(iii) again we have $\mathsf{coat}(c(x_2)) = \{x_2, c(x_1)\}$ and hence $c(x_2) \in D$, so that $[x, c(x_2)] \subseteq D$. If $D = [x, c(x_2)]$ we are done, and it is now clear how the general case will be reached by induction. □

Corollary B.6. *If P is completely compressible and $K_{2,3}$-avoiding, then all the $D^P(a, b)$ are closed intervals.*

References

[1] F. Brenti, The intersection cohomology of Schubert varieties is a combinatorial invariant, European J. Combin. 25 (2004) 1151–1167.
[2] F. Brenti, F. Caselli, M. Marietti, Special matchings and Kazhdan–Lusztig polynomials, Adv. Math. 2 (2006) 555–601.
[3] F. Brenti, F. Caselli, M. Marietti, Diamonds and Hecke algebra representations, Int. Math. Res. Not. (2006) 34. Art. ID. 29407.
[4] E. Delanoy, Combinatorial invariance of Kazhdan–Lusztig polynomials on intervals starting from the identity, J. Algebraic Combin. 24 (2006) 437–463.
[5] E. Delanoy, Définition combinatoire des polynômes de Kazhdan–Lusztig, Doctoral Thesis, Université Lyon 1, 2006.
[6] V.V. Deodhar, Some characterizations of the Bruhat ordering of a Coxeter group and determination of the relative Möbius function, Invent. Math. 39 (1977) 187–198.
[7] F. du Cloux, An abstract model for Bruhat intervals, European J. Combin. 21 (2000) 197–222.
[8] F. du Cloux, Coxeter, version beta. Available on: http://www.desargues.univ-lyon1.fr/home/ducloux/coxeter.html.
[9] F. du Cloux, Rigidity of Schubert closures and invariance of Kazhdan–Lusztig polynomials, Adv. Math. 180 (2003) 146–175.
[10] M. Dyer, Hecke algebras and reflections in Coxeter groups, Ph.D. Thesis, University of Sydney, 1987.
[11] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990.
[12] J. Stembridge, On the fully commutative elements of Coxeter groups, J. Algebraic Combin. 5 (1996) 353–385.